Two problems of statistical estimation for stochastic processes

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Outline

Inhomogeneous Poisson process

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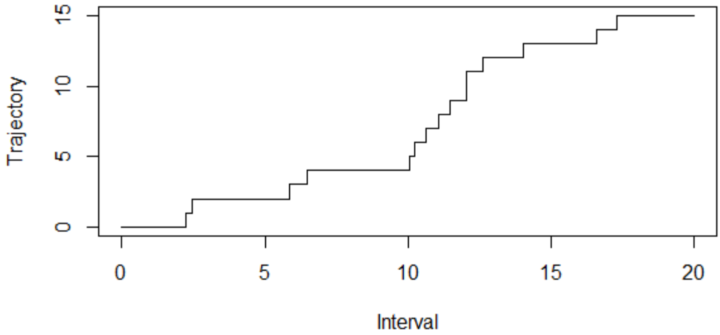
- $X(0) = 0$, has independent increments and there exists a non-negative, increasing function $\Lambda(t)$ s.t. for all $t \in [0, T]$

$$\mathbf{P}(X(t) = k) = \frac{[\Lambda(t)]^k}{k!} e^{-\Lambda(t)},\ k = 0, 1, \cdots .$$

Trajectory of a Poisson (counting) process



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- The non-negative function $\lambda(\cdot)$ is called the intensity function and the periodicity of a Poisson process means the periodicity of its intensity function

$$\lambda(t)=\lambda(t+k\tau),\ t\in[0,\tau],\ k\in\mathcal{Z}_+.$$

Mean and the Intensity functions

- With the notations

$$X_j(t) = X((j-1)\tau + t) - X((j-1)\tau),\ t \in [0,\tau],$$
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- We would like to have Hájek-Le Cam type lower bounds for function estimation.

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- The simplest estimator is the *empirical mean function*

$$\hat{\Lambda}_n(t) = \frac{1}{n} \sum_{j=1}^n X_j(t),\ t \in [0, \tau].$$

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- Can we have other asymptotically efficient estimators?

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- Demanding existence of derivatives of higher order of the unknown function, we can enlarge the class of as. efficient estimators.

First results

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- Kernels satisfy

$$K_n(u) \geq 0,\ u \in \left[-\frac{\tau}{2}, \frac{\tau}{2}\right],\ \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} K_n(u)\mathrm{d}u = 1,\ n \in \mathcal{N},$$

and we continue them τ periodically on the whole real line **R**

$$K_n(u) = K_n(-u),\quad K_n(u) = K_n(u + k\tau),\ u \in \left[-\frac{\tau}{2}, \frac{\tau}{2}\right],\ k \in \mathcal{Z}.$$

Kernel-type estimator

- Consider the trigonometric basis in $\mathrm{L}_2[0,\tau]$

$$\phi_1(t)=\sqrt{\frac{1}{\tau}},\ \phi_{2l}(t)=\sqrt{\frac{2}{\tau}}\cos\frac{2\pi l}{\tau}t,\ \phi_{2l+1}(t)=\sqrt{\frac{2}{\tau}}\sin\frac{2\pi l}{\tau}t.$$

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- Coefficients of the kernel-type estimator w.r.t. this basis

$$\tilde{\Lambda}_{1,n}=\hat{\Lambda}_{1,n},\ \tilde{\Lambda}_{2l,n}=\sqrt{\frac{\tau}{2}}K_{2l,n}(\hat{\Lambda}_{2l,n}-\Lambda^*_{2l})+\Lambda^*_{2l},$$

$$\tilde{\Lambda}_{2l+1,n}=\sqrt{\frac{\tau}{2}}K_{2l,n}(\hat{\Lambda}_{2l+1,n}-\Lambda^*_{2l+1})+\Lambda^*_{2l+1},\ l\in\mathcal{N},$$

where $\hat{\Lambda}_{l,n}$ are the Fourier coefficients of the EMF.

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$$\lim_{n\to+\infty} \sup_{\Lambda\in\mathcal{B}(R)} \left(\mathbf{E}_\Lambda || \sqrt{n}(\tilde{\Lambda}_n - \Lambda)||^2 - \int_0^\tau \Lambda(t)\mathrm{d}t \right) = 0.$$

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- The kernel-type estimator is asymptotically efficient over $\Sigma(R)$

$$\lim_{n \to +\infty} \sup_{\Lambda \in \Sigma(R)} \left(\mathbf{E}_\Lambda ||\sqrt{n}(\tilde{\Lambda}_n - \Lambda)||^2 - \int_0^\tau \Lambda(t) \mathrm{d}t \right) = 0.$$

Example of another as. effective estimator

- Consider a kernel

$$K(u) \geq 0,\ u \in \left[-\frac{\tau}{2}, \frac{\tau}{2}\right],\ \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} K(u)\mathrm{d}u = 1,$$

$$K(u) = K(-u), \quad K(u) = K(u + k\tau),\ u \in \left[-\frac{\tau}{2}, \frac{\tau}{2}\right],\ k \in \mathcal{Z}.$$

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- Calculate the constant C.

Related works

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- Asymptotic efficiency in non-parametric estimation problems was done for the first time in [Pinsker M.S., 1980],
- where the analogue of the inverse of the Fisher information in non-parametric estimation problem was calculated (Pinsker's constant).

Main theorems (joint work with Yu.A. Kutoyants)

- Introduce

$$\mathcal{F}_m^{per}(R,S)=\left\{\Lambda(\cdot):\int_0^\tau[\lambda^{(m-1)}(t)]^2\mathrm{d}t\leq R,\,\Lambda(0)=0,\,\Lambda(\tau)=S\right\}$$

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where $R>0$, $S>0$, $m>1$, are given constants.

- For all estimators $\bar{\Lambda}_n(t)$ of the mean function $\Lambda(t)$, following lower bound holds

$$\varliminf_{n\to+\infty}\sup_{\Lambda\in\mathcal{F}_m^{per}(R,S)} n^{\frac{1}{2m-1}}\left(\mathbf{E}_\Lambda||\sqrt{n}(\bar{\Lambda}_n-\Lambda)||^2-\int_0^\tau\Lambda(t)\mathrm{d}t\right)\geq-\Pi,$$

Main theorems (joint work with Yu.A. Kutoyants)

- Introduce

$$\mathcal{F}_m^{per}(R,S)=\left\{\Lambda(\cdot):\ \int_0^\tau [\lambda^{(m-1)}(t)]^2\mathrm{d}t\leq R,\ \Lambda(0)=0,\ \Lambda(\tau)=S\right\}$$

where $R>0$, $S>0$, $m>1$, are given constants.

- For all estimators $\bar\Lambda_n(t)$ of the mean function $\Lambda(t)$, following lower bound holds

$$\varliminf_{n\to+\infty}\ \sup_{\Lambda\in\mathcal{F}_m^{per}(R,S)} n^{\frac{1}{2m-1}}\left(\mathbf{E}_\Lambda||\sqrt{n}(\bar\Lambda_n-\Lambda)||^2-\int_0^\tau\Lambda(t)\mathrm{d}t\right)\geq-\Pi,$$

- where

$$\Pi=\Pi_m(R,S)=(2m-1)R\left(\frac{S}{\pi R}\frac{m}{(2m-1)(m-1)}\right)^{\frac{2m}{2m-1}},$$

plays the role of the Pinsker's constant.

Second order as. efficient estimator

- Consider

$$\Lambda_n^*(t) = \hat{\Lambda}_{0,n}\phi_0(t) + \sum_{l=1}^{N_n} \tilde{K}_{l,n}\hat{\Lambda}_{l,n}\phi_l(t),$$

where $\{\phi_l\}_{l=0}^{+\infty}$ is the trigonometric cosine basis, $\hat{\Lambda}_{l,n}$ are the Fourier coefficients of the EMF w.r.t. this basis and

$$\tilde{K}_{l,n} = \left(1 - \left|\frac{\pi l}{\tau}\right|^m \alpha_n^*\right)_+, \quad \alpha_n^* = \left[\frac{S}{nR}\frac{\tau}{\pi}\frac{m}{(2m-1)(m-1)}\right]^{\frac{m}{2m-1}},$$

$$N_n = \frac{\tau}{\pi}(\alpha_n^*)^{-\frac{1}{m}} \approx \mathrm{C} n^{\frac{1}{2m-1}}, \quad x_+ = \max(x,0),\ x \in \mathbf{R}.$$

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- The estimator $\Lambda_n^*(t)$ attains the lower bound described above, that is,

$$\lim_{n\to+\infty} \sup_{\Lambda\in\mathcal{F}_m(R,S)} n^{\frac{1}{2m-1}} \left(\mathbf{E}_\Lambda||\sqrt{n}(\bar{\Lambda}_n - \Lambda)||^2 - \int_0^\tau \Lambda(t)\mathrm{d}t\right) = -\Pi.$$

Sketch of the proof (Lower bound)

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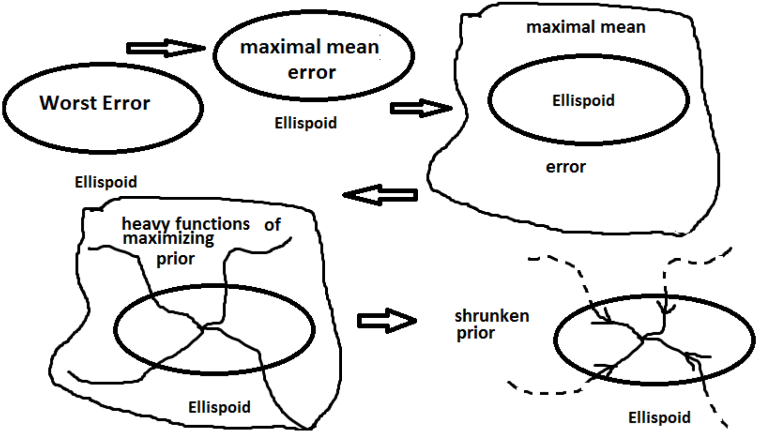
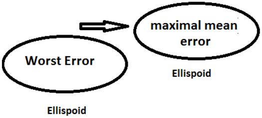
  - 

First step



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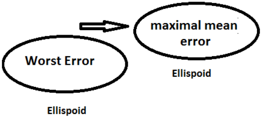
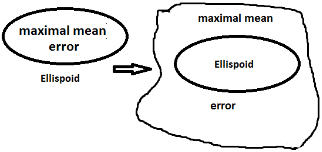


- Reduce the minimax problem to a Bayes risk maximization problem

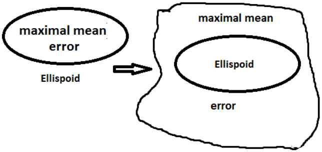
$$\sup_{\Lambda\in\mathcal{F}_m^{(per)}(R,S)} \left(\mathbf{E}_\Lambda||\bar{\Lambda}_n-\Lambda||^2-\mathbf{E}_\Lambda||\hat{\Lambda}_n-\Lambda||^2\right)\geq$$

$$\sup_{\mathbf{Q}\in\mathcal{P}}\int_{\mathcal{F}_m^{(per)}(R,S)}\left(\mathbf{E}_\Lambda||\bar{\Lambda}_n-\Lambda||^2-\mathbf{E}_\Lambda||\hat{\Lambda}_n-\Lambda||^2\right)\mathrm{d}\mathbf{Q}.$$

Second step


maximal mean
error
Ellispoid
maximal mean
Ellispoid
error

Second step



- In the maximization problem replace the set of probabilities $\mathcal{P}(\mathcal{F})$ concentrated on $\mathcal{F}_m^{(per)}(R,S)$

$$\sup_{\mathbf{Q}\in\mathcal{P}(\mathcal{F})} \int_{\mathcal{F}_m^{(per)}(R,S)} \left(\mathbf{E}_\Lambda||\bar{\Lambda}_n-\Lambda||^2-\mathbf{E}_\Lambda||\hat{\Lambda}_n-\Lambda||^2\right) \mathrm{d}\mathbf{Q},$$

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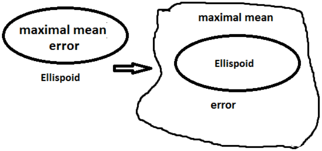

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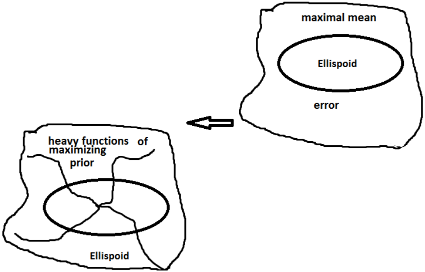
- by the set of probabilities $\mathbf{E}(\mathcal{F})$ concentrated on $\mathcal{F}_m^{(per)}(R,S)$ in mean.

Third step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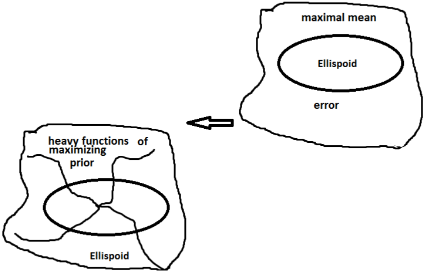
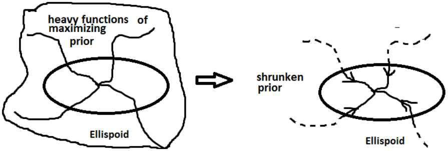
- 

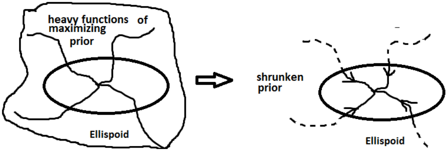
- Replace the ellipsoid by the least favorable parametric family (heavy functions)

$$\sup_{\Lambda_\theta \in \mathcal{F}_m^{(per)}(R,S)} \int_\Theta \left(\mathbf{E}_\theta ||\bar{\Lambda}_n - \Lambda_\theta||^2 - \mathbf{E}_\theta ||\hat{\Lambda}_n - \Lambda_\theta||^2 \right) \mathrm{d}\mathbf{Q}.$$

Fourth step


heavy functions of maximizing prior
Ellispoid
shrunken prior
Ellispoid

Fourth step



- Shrink the heavy functions and the least favorable prior distribution to fit the ellipsoid

$$\mathbf{Q}\{\theta : \Lambda_\theta \notin \mathcal{F}_m^{(per)}(R,S)\} = o(n^{-2}).$$

Sketch of the proof (Upper bound)

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- Since we demanded existence of the $m-$th derivative, then the class of as. efficient estimators is

$$n \sup_{l\geq 1} \left| \frac{K_{l,n} - 1}{\left(\frac{\pi l}{\tau}\right)^{2m}} \right|^2 \longrightarrow 0,$$

Sketch of the proof (Upper bound) 2

- Consider the class

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$$\alpha_n^* = \left[\frac{S}{nR} \frac{\tau}{\pi} \frac{m}{(2m-1)(m-1)} \right]^{\frac{m}{2m-1}} \approx \mathrm{C} n^{-\frac{m}{2m-1}} .$$

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- (the condition $n(\alpha_n^*)^2 = \mathrm{C} n^{-\frac{1}{2m-1}} \longrightarrow 0$ is also satisfied).

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- Adaptive estimation: construct an estimator that does not depend on m, S, R.
- Consider other models or formulate a general result for non-parametric LAN.

Forward-Backward SDE

- For a given (*Forward*) Stochastic Differential Equation (SDE)

$$\mathrm{d}X_t = S(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad X_0 = x_0,\ t \in [0, T],$$

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  and functions $f(t, x, y, z), \Phi(x)$,
- we are searching a couple of processes (Y_t, Z_t)
- which satisfies the stochastic differential equation

$$\mathrm{d}Y_t = -f(t, X_t, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t, \; 0 \le t \le T,$$

  with the terminal condition $Y_T = \Phi(X_T)$.
  [Pardoux, Peng, 1992].

Solution of a FBSDE

- If the function $u = u(t, x)$ satisfies the following partial differential equation

$$\frac{\partial u}{\partial t} + S(t,x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 u}{\partial x^2} = -f\left(t, x, u, \sigma(t,x)\frac{\partial u}{\partial x}\right),$$

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- then the processes

$$Y_t = u(t, X_t),\ Z_t = \sigma(t, X_t)u'_x(t, X_t),$$

satisfy (by the Itô's formula) the following SDE

$$\mathrm{d}Y_t = \left[\frac{\partial u}{\partial t}(t,X_t) + S(t,X_t)\frac{\partial u}{\partial x}(t,X_t) + \frac{1}{2}\sigma^2(t,X_t)\frac{\partial^2 u}{\partial x^2}(t,X_t)\right]\mathrm{d}t$$
$$+ \sigma(t,X_t)\frac{\partial u}{\partial x}(t,X_t)\mathrm{d}W_t,$$

hence $\mathrm{d}Y_t = -f(t, X_t, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t, \quad Y_T = \Phi(X_T).$

Statistical Estimation

- Suppose that the coefficient $\sigma(\cdot)$ of the forward equation depends on an unknown parameter $\vartheta \in \Theta = (\alpha, \beta)$,

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- We have discrete time observations of the solution $\{X_t, \ t \in [0, T]\}$ of the forward SDE

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- Our goal is, based on this observations, estimate the solution of the FBSDE

$$Y_t = u(\vartheta, t, X_t), \quad Z_t = \sigma(\vartheta, t, X_t)u'_x(\vartheta, t, X_t).$$

which also depends on the unknown parameter.

Heuristics

- We have to construct an estimator $\vartheta^*_{t,n}$ of ϑ, based on the observations up to time t

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which

1. have to be easily calculable for every t,
2. the estimator $Y^*_{t,n} = u(\vartheta^*_{t,n}, t, X_{t_k})$ must have asymptotically the smallest quadratic error

$$\mathbf{E}_\vartheta (Y^*_{t,n} - Y_t)^2 \longrightarrow \min, \quad n \longrightarrow +\infty.$$

Parameter in Diffusion

- How to construct a consistent estimator for the unknown parameter in the diffusion coefficient of a SDE?

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- [Genon-Catalot, Jacod, 1993] We are considering the minimum contrast estimator

$$\hat{\vartheta}_{t,n} = \arg\min_{\vartheta \in \Theta} U_k(\vartheta),$$

where the contrast function is

$$U_k(\vartheta) = \sum_{j=1}^{k} \left[\delta \ln \sigma^2(\vartheta, t_{j-1}, X_{t_{j-1}}) + \frac{(X_{t_j} - X_{t_{j-1}})^2}{\sigma^2(\vartheta, t_{j-1}, X_{t_{j-1}})} \right].$$

- This estimator is consistent and asymptotically mixed normal

$$\hat{\vartheta}_{t,n} \longrightarrow \vartheta_0, \quad \delta^{-1/2}(\hat{\vartheta}_{t,n} - \vartheta_0) \Longrightarrow \xi_t(\vartheta_0),$$

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$$\mathrm{I}_t(\vartheta_0) = 2 \int_0^t \frac{\dot{\sigma}(\vartheta_0, s, X_s)^2}{\sigma(\vartheta_0, s, X_s)^2} \mathrm{d}s,$$

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- Hence, $\hat{Y}_{t,n} = u(\hat{\vartheta}_{t,n}, t, X_{t_k})$ will be a consistent estimator for the process Y_t for each fixed $t \in (0, T]$.
- But this estimator has a problem: **it is difficult to calculate.**

Estimation of the solution of a FBSDE (joint work with Yu. A. Kutoyants)

Example: Consider the following SDE

$$\mathrm{d}X_t = S(X_t)\mathrm{d}t + \sqrt{\vartheta}h(X_t)\mathrm{d}W_t, \quad X_0 = 0.$$

Suppose the problem is to estimate the unknown positive parameter ϑ based on continuous time observations $\{X_t,\, t \in [0, T]\}$. It is well known that by continuous time observations the unknown parameter in the diffusion coefficient can be estimated without an error.

Estimation without an Error

- Applying the Itô formula to the function $G(x) = x^2$, we get

$$X_t^2 = 2\int_0^t X_s \mathrm{d}X_s + \vartheta \int_0^t h^2(X_s)\mathrm{d}s,$$

hence, with the probability 1

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- Applying the Itô formula to the function $G(x) = x^2$, we get

$$X_t^2 = 2\int_0^t X_s \mathrm{d}X_s + \vartheta\int_0^t h^2(X_s)\mathrm{d}s,$$

hence, with the probability 1

- $$\vartheta = \frac{X_t^2 - 2\int_0^t X_s \mathrm{d}X_s}{\int_0^t h^2(X_s)\mathrm{d}s}, \quad \forall 0 < t \leq T.$$

Estimation without an Error

- Applying the Itô formula to the function $G(x) = x^2$, we get

$$X_t^2 = 2\int_0^t X_s \mathrm{d}X_s + \vartheta \int_0^t h^2(X_s)\mathrm{d}s,$$

  hence, with the probability 1

- $$\vartheta = \frac{X_t^2 - 2\int_0^t X_s \mathrm{d}X_s}{\int_0^t h^2(X_s)\mathrm{d}s}, \quad \forall 0 < t \leq T.$$

- So, if one has continuous time observations even in very small interval, then the unknown parameter can be found for almost all realizations.

The First Estimator

Suppose that $\tau > 0$ is a small number and we have to estimate the solution Y_t at the points $t \in [\tau, T]$. Denote $m = \left[\frac{\tau}{T} n\right]$, that is

$$0 = t_0 < \cdots < t_m \leq \tau < t_{m+1} < \cdots < t_k < t < t_{k+1} < \cdots < t_n = T.$$

Consider

$$\tilde{\vartheta}_{\tau,n} = \arg\min_{\vartheta \in \Theta} U_m(\vartheta),$$

where $U_m(\vartheta)$ is the same contrast function. Then, by previous arguments about estimation without an error, we have that the estimator

$$\tilde{Y}_{t,n} = u(\tilde{\vartheta}_{\tau,n}, t, X_{t_k})$$

is consistent. This estimator is the simplest one but it is not asymptotically efficient.

Efficiency

Before finding the best estimator we have to define what is the best estimator. [Ibragimov, Hasminskii, 1981], [Kutoyants, 2004] For this we need a theorem that compares all estimators as $n \longrightarrow +\infty$. We suppose that the function $u(\vartheta, t, X_t)$ is sufficiently smooth w.r.t. ϑ. Our first result is

Theorem (Lower bound)

For all estimators $\bar{Y}_{t_k,n}$ of the process Y_t the following inequality holds

$$\underline{\lim}_{\varepsilon\to 0}\ \underline{\lim}_{n\to+\infty}\ \sup_{|\vartheta-\vartheta_0|<\varepsilon} \mathbf{E}_\vartheta \ell\left(\delta^{-1/2}\left(\bar{Y}_{t_k,n}-Y_{t_k}\right)\right) \geq$$
$$\geq \mathbf{E}_{\vartheta_0}\ell(\dot{u}(\vartheta_0,t,X_t)\xi_t(\theta_0)), \quad \ell(u)=|u|^p, p>0.$$

Remark. Here we compare the estimators with the unknown process in the point t_k instead of point t.

Ideas of the Proof.

The proof of this theorem based on the fact [Dohnal, 1987], [Gobet, 2001] that under some regularity conditions the probability measures $\{\mathcal{P}_{\vartheta}^{n,k}, \vartheta \in \Theta\}$ induced by the observations

$$\mathbf{X}^k = (X_{t_0}, X_{t_1}, \cdots, X_{t_k})$$

$$\mathcal{P}_{\vartheta_0}^{n,k}(B) = \mathcal{P}(\mathbf{X}^k \in B), \forall B \in \mathcal{B}(\mathbf{R}^k)$$

satisfy the Local Asymptotic Mixed Normality (LAMN) [Jeganathan, 1982] condition: for all $\vartheta_0 \in \Theta$,

$$\ln \frac{\mathrm{d}\mathcal{P}_{\vartheta_0+\sqrt{\delta}v}^{n,k}}{\mathrm{d}\mathcal{P}_{\vartheta_0}^{n,k}}(\mathbf{X}^k) = v\Delta_{n,k}(\vartheta_0) - \frac{1}{2}v^2\mathrm{I}_{n,k}(\vartheta_0) + r_{n,k}(v,\vartheta_0),$$

where $r_{n,k}(v,\vartheta_0) \longrightarrow 0$, in $\mathcal{P}_{\vartheta_0}^{n,k}$ probability, for every $v \in R$,

$$\Delta_{n,k}(\vartheta_0) = \sqrt{2}\sum_{j=1}^{k} \frac{\dot\sigma(\vartheta_0, t_{j-1}, X_{t_{j-1}})}{\sigma(\vartheta_0, t_{j-1}, X_{t_{j-1}})^2}(X_{t_j} - X_{t_{j-1}} - S(t_{j-1}, X_{t_{j-1}})\delta)$$

$$\Longrightarrow \Delta_t(\vartheta_0) = \sqrt{2}\int_0^t \frac{\dot\sigma(\theta_0, s, X_s)}{\sigma(\theta_0, s, X_s)} dw(s),$$

stably in $\mathcal{P}_{\vartheta_0}^{n,k}$ law, $w(t)$ is a Wiener process independent from $\{X_s, 0 \leq s \leq t\}$ and, in $\mathcal{P}_{\vartheta_0}^{n,k}$ probability,

$$\mathrm{I}_{n,k}(\vartheta_0) = 2\sum_{j=1}^{k} \frac{\dot\sigma(\vartheta_0, t_{j-1}, X_{t_{j-1}})^2}{\sigma(\vartheta_0, t_{j-1}, X_{t_{j-1}})^2}\delta$$

$$\longrightarrow \mathrm{I}_t(\vartheta_0) = 2\int_0^t \frac{\dot\sigma(\vartheta_0, s, X_s)^2}{\sigma(\vartheta_0, s, X_s)^2} \mathrm{d}s,$$

$$\xi_t(\theta_0) = \frac{\Delta_t(\vartheta_0)}{\mathrm{I}_t(\vartheta_0)}.$$

Asymptotically Efficient Estimator

To find asymptotically efficient estimator we use the idea of **Le Cam**'s one-step Maximum Likelihood Estimator (MLE). For FBSDEs this idea was first implemented in the paper [Kutoyants, Zhou, 2014].

We take the preliminary estimator $\tilde{\vartheta}_{\tau,n}$ constructed by the observations

$$0 = t_0 < t_1 < \cdots < t_m \leq \tau$$

and improve it in the following way

$$\vartheta^*_{t,n} = \tilde{\vartheta}_{\tau,n} + \sqrt{\delta}\frac{\Delta_{n,k}(\tilde{\vartheta}_{\tau,n})}{\mathrm{I}_{n,k}(\tilde{\vartheta}_{\tau,n})}.$$

The Second Estimator

Then, the estimator $Y^*_{t,n} = u(\vartheta^*_{t,n}, t, X_{t_k})$ is consistent and asymptotically mixed normal

$$Y^*_{t_k,n} \longrightarrow Y_t, \quad \delta^{-\frac{1}{2}}(Y^*_{t_k,n} - Y_{t_k}) \Longrightarrow \dot u(\vartheta_0, t, X_t)\xi_t(\theta_0),$$

as $n \longrightarrow +\infty$ and

Theorem

The estimator $Y^*_{t,n} = u(\vartheta^*_{t,n}, t, X_{t_k})$ is asymptotically efficient

$$\lim_{\varepsilon\to 0}\lim_{n\to+\infty}\sup_{|\vartheta-\vartheta_0|<\varepsilon} \mathbf{E}_\vartheta \ell\left(\delta^{-1/2}\left(Y^*_{t_k,n} - Y_{t_k}\right)\right) =$$

$$= \mathbf{E}_{\vartheta_0}\ell(\dot u(\vartheta_0, t, X_t)\xi_t(\theta_0)), \quad \ell(u) = |u|^p, p > 0.$$

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It's Over!